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Topological Partition Function and String-String Duality

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Abstract

The evidence for string/string-duality can be extended from the matching of the vector couplings to gravitational couplings. In this note this is shown in the rank three example, the closest stringy analog of the Seiberg/Witten-setup, which is related to the Calabi-Yau $WP^4_{1,1,2,2,6}(12)$. I provide an exact analytical verification of a relation checked by coefficient comparison to fourth order by Kaplunovsky, Louis and Theisen.

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Recent progress in nonperturbative understanding of string theory is based on the conjectured string-string duality [1],[2] first proposed in $D=6$ for the heterotic string on T^4 and the type IIA string on $K3$, then extended and partially verified in $D=4$ for the heterotic string on $K3\times T^2$ and type IIA on a Calabi-Yau [3],[4],[5],[6]. In the course of accumulating evidence for the conjectured equivalence of $N=2$ string theories in [9] new material was added in comparing the holomorphic F_1-functions describing 1-loop gravitional couplings of vectormultiplets in the proposed [4] dual pair consisting of the heterotic string on $K3\times T^2$ with gauge group $U(1)^3$ (graviphoton, vector partner of the dilaton and of the toroidal modulus T; the second toroidal modulus U is locked at $U=T$) and the type IIA string on a suitable Calabi-Yau (cp. also [7],[8]). Concretely Kaplunovsky, Louis and Theisen [9] showed for the rank three model of [4] that

$$F_1^{het}=\frac{6}{4\pi^2}\log[y^{-2}(j(t)-j(i))^{\frac{1}{6}}\eta^2(t)^{-50}]\ ,$$

where[1]

$$\begin{aligned} t &= t_1=iT \\ y &= e^{-8\pi^2 S^{inv}}=g(iT)e^{-8\pi^2 S} \end{aligned}$$

with t_1 and $t_2=4\pi iS$ the special coordinates (the complexified Kähler class is related to the cohomology classes of the relevant divisors (cp. [12]) by $B+iJ=t_1H+t_2L$) of $WP^4_{1,1,2,2,6}(12)^{-252}_{2,128}$, which are here already matched with their heterotic counterparts. The mirror map for the complex structure deformations of the mirror Calabi-Yau coming from

$$p=z_1^{12}+z_2^{12}+z_3^6+z_4^6+z_5^2+a_0z_1z_2z_3z_4z_5+a_1z_1^6z_2^6$$

is given by $(q_j=e^{2\pi it_j})$

$$x=j(q_1)^{-1}+O(q_2), y=g(q_1)q_2+O(q_2^2)$$

(with uniformizing variables at large complex structure $x=a_1a_0^{-6}$,$y=a_1^{-2})^2$.
At weak coupling the first four terms in q_1 are then matched [9] by comparing to a series expansion for F_1^{II}.
I will show here exact agreement in q_1 (at weak coupling). Let us first reformulate the heterotic result as

$$F_1^{het}\sim\log[y^{-\alpha}(j(t)-j(i))^{\beta}\eta^2(t)^{-\gamma}]\ ,$$

[1]We use their identification of the modular invariant dilaton(cp. also [10]) got by matching the two prepotentials.

[2]The parameters of [9] and [12] are related by $a_0=-12\psi, a_1=-2\phi$.

where in

$$\begin{aligned} \alpha &= 12\beta \\ \gamma &= 300\beta \end{aligned}$$

the 12 (the 24 in equations (18), (22) of [9]) comes from curvature coupling normalization and the 300 is $b_{grav} = 48 - \chi$ [9].
I will show first the following on the type II side (for some numbers α, β, γ, c)

$$F_1^{II} = \log[y^{-\alpha} E_4^{\frac{c-1}{2}}(t)(j(t) - j(i))^{\beta} \eta^2(t)^{-\gamma}]$$

(up to an additive constant) with $\gamma = 2(c + 3 - \frac{\chi}{12})$. By the holomorphic anomaly ofF_1 [11] we have [12]

$$F_1^{II} = \log[(\frac{12\psi}{\omega_0})^{5-\frac{\chi}{12}} \frac{\partial(\psi, \chi)}{\partial(t_1, t_2)} f]$$

with $f = \Delta^a (\phi^2 - 1)^b \psi^c$, $\Delta = (\frac{1728}{2}\psi^6 + \phi)^2 - 1$, where $a = -\frac{1}{6}, b = -\frac{2}{3}, c = 1$ and ω_0 the fundamental period (note that $\omega_0^2|_{y=0} = E_4$). Now at weak coupling we have $\frac{\partial(x,y)}{\partial(q_1,q_2)} \sim \frac{j_q g}{j^2}(q_1)$, so that

$$\frac{\partial(\psi, \phi)}{\partial(t_1, t_2)} \sim y^{-1} \psi^{-5} j_\tau,$$

which together with (remember $j = \frac{E_4^3}{\eta^{24}}$)

$$\begin{aligned} \psi &\sim (\frac{1}{yx^2})^{\frac{1}{12}} = y^{-\frac{1}{12}} \frac{E_4^{\frac{1}{2}}}{\eta^4} \\ \Delta|_{y\approx 0} &\sim y^{-1}(j - j(i))^2 \end{aligned}$$

and with the relation $j_\tau^2 \sim j(j - j(i))E_4$ leads to the announced expression of F_1^{II} with

$$\begin{aligned} \alpha &= 1 + a + b + \frac{1}{12}(c - \frac{\chi}{12}) \\ \beta &= 2a + \frac{1}{2} \\ \gamma &= 2(c + 3 - \frac{\chi}{12}) \,. \end{aligned}$$

Now one actually has $c = 1$ [12]. Then , on the one hand, insertion of the values $a = -\frac{1}{6}, b = -\frac{2}{3}$ obtained by comparing at the large radius limit with the topological intersection numbers [12] leads immediately to the correct *explicit* powers of α, β, γ. On the other hand one can also get the *abstract cohomological meaning* of $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\beta}$ on the Calabi-Yau-side (in our two parameter model) by noting the possibility of direct

comparison of the expression for F_1^{II} with the large radius limit (leaving aside the intermediate step of determination of a and b) obtaining the result

$$\alpha = \frac{1}{12}c_2 \cdot L = \frac{1}{12}c_2(L) = \frac{1}{12}\chi_{K3}$$

and

$$\begin{aligned} \beta + \frac{\gamma}{12} &= \frac{1}{12}c_2 \cdot H \\ &= \frac{1}{12}[2\chi_{K3} + c_2(E) - E^3] \\ &= \frac{1}{12}[2\chi_{K3} + 2\chi_C - 4\chi_C] \\ &= \frac{1}{12}2(\chi_{K3} - \chi_C), \end{aligned}$$

(where we used the relation $E^3 = -8 = 4\chi_C$ between the singular curve C of the Calabi-Yau and the ruled surface E of its pointwise resolution [12]), so $12\beta + \gamma = 2(4 - \frac{\chi}{12})(12\frac{\beta}{\gamma} + 1)$ equals[3] $2(\chi_{K_3} - \chi_C) = 2(4 - \frac{\chi}{12} + 1)$ from which it follows that $2(4 - \frac{\chi}{12})12\frac{\beta}{\gamma} = 2 \cdot 1$, which shows that

$$\begin{aligned} \frac{\gamma}{\beta} &= 12(4 - \frac{\chi}{12}) \\ \frac{\alpha}{\beta} &= \frac{\chi_{K3}}{12}\frac{12(4 - \frac{\chi}{12})}{\gamma} = \frac{\chi_{K3}}{2}. \end{aligned}$$

This agrees with the meaning of the numbers on the heterotic side.

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[3] Note that because of the K_3-fibration $\chi = (\chi_{P1} - 12)\chi_{K3} + 12(\chi_C + 1)$ so $-\frac{\chi}{12} = -4 + \chi_{K_3} - \chi_C - 1$ (the $+1$ comes from the possibility of having $z_3 = z_4 = z_5 = 0$).

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